

Classical conformal blocks

Pietro Menotti

Dipartimento di Fisica, Università di Pisa

Largo B. Pontecorvo 3, I-56127, Pisa, Italy

e-mail: pietro.menotti@unipi.it

Abstract

We give a simple iterative procedure to compute the classical conformal blocks on the sphere to all order in the modulus.

1 Introduction

A lot of work has been devoted about the structure of conformal blocks in conformal field theories, and in particular in quantum Liouville theory. There is no closed expression for such conformal blocks which are formally defined as a power series in the modulus x and there exist recursive procedures to compute quantum conformal blocks (see e.g. [1, 2, 3, 4, 5, 6]).

A remarkable conjecture has been put forward in [3] about the exponentiation of such blocks in the semiclassical limit giving rise to the so-called classical conformal blocks. Moreover such classical conformal blocks have been shown to be connected to the accessory parameters appearing in the auxiliary differential equation through a simple relation [3, 7, 8].

As a rule classical conformal blocks are obtained from the $b \rightarrow 0$ limit of the quantum conformal blocks. In such limit heavy cancellations occur in the quantum expression which give rise to the above mentioned exponentiation. No general proof appears to be available of such exponentiation process even if it has been checked to a few orders in the expansion in x .

In [9] the problem has been addressed of determining the accessory parameters directly from the auxiliary equation and the monodromy condition without appeal to the quantum conformal blocks. The method was applied to the computation of the first two non trivial terms in the expansion of the accessory parameter on the sphere in the modulus x .

The procedure was to deform the contour in the complex plane embracing the singularities at the origin and at the point x , to a contour which embraces the cut from 1 to infinity and closes through a circle at infinity. The advantage of such a procedure was that one dealt with an expansion of the energy momentum tensor which converges for $|x| < 1$ and that the asymptotic behavior of the solutions of the accessory equation could be computed through perturbation theory. Such perturbative series in x was computed up to second order and the result compared with success with the semiclassical limit of the quantum conformal block.

An approach similar in nature was applied in [10, 11] to compute 5-point classical conformal blocks for $b \rightarrow 0$ when some of the intervening charges become heavy and others stay light. Similar limits have been considered in [12, 4, 13, 14].

In the present paper it is shown how the method of [9] can be extended to provide a very simple algebraic iterative method to compute the accessory parameter to any order in x . We apply the method explicitly to the third order in x but the iterative procedure can be carried on with great ease to any order. In fact at each step it boils down to the solution

of a linear equation. In [7] and in [15], using techniques developed in [16], the conformal blocks to first and second order have been explicitly written and in [5] the explicit (and rather long) form of the quantum conformal blocks has been given up to the third order included. We compare such values with our results finding complete agreement.

2 The expansion of the accessory parameters

In this section we extend the procedure of paper [9] to all order in the powers of the modulus. In addition the procedure is drastically simplified.

To make the paper more self contained we repeat some elements explained in [9]. The notation is the same as the one adopted in [9].

We start from the auxiliary differential equation of the 4-point Liouville problem given by

$$y''(z) + Q(z)y(z) = 0 \quad (1)$$

with

$$Q = \frac{\delta_0}{z^2} + \frac{\delta}{(z-x)^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta - \delta_1}{z(z-1)} - \frac{C(x)}{z(z-x)(z-1)} \quad (2)$$

where $\delta_j = (1 - \lambda_j^2)/4$ and $C(x)$ is the accessory parameter. The generalized monodromy problem [15] is to fix the accessory parameter $C(x)$ by the requirement that the monodromy M along a contour encircling both 0 and x has a fixed trace $-2 \cos \pi \lambda_\nu$. The idea is to use a transformation of the variable z which mimics the transformation induced by the Virasoro generators. Then by deforming the monodromy contour as in paper [9] we find a simple iterative procedure which allow to compute the $C(x)$ as a power expansion in x to all orders.

Our $C(x)$ is related to the parameter used in [3, 15] which we shall call $C_L(x)$ by $C(x) = x(1-x)C_L(x)$.

We supply the explicit results up to $C'''(0)$ even if it is very simple to proceed to any order. We compare the obtained results with the values of $C''(0)$ and $C'''(0)$ given in [7, 15] and to the expression for $C'''(0)$ derived from [5] finding complete agreement.

At $x = 0$ the the requirement $\text{tr} M = -2 \cos \pi \lambda_\nu$ fixes the value of $C(0)$

$$C(0) = \delta_\nu - \delta_0 - \delta \quad (3)$$

We write

$$Q(x) = Q_0 + xQ_1 + x^2Q_2 + \dots \quad (4)$$

with

$$Q_0 = \frac{\delta_\nu}{z^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_1 - \delta_\nu}{z(z-1)} \quad (5)$$

$$Q_1 = \frac{2\delta - C'(0)}{z^2(z-1)} - \frac{2\delta + C(0)}{z^3(z-1)} \quad (6)$$

$$Q_2 = -\frac{C''(0)}{2z^2(z-1)} + \frac{3\delta - C'(0)}{z^3(z-1)} - \frac{3\delta + C(0)}{z^4(z-1)} \quad (7)$$

and in general

$$Q_n = \frac{Q^{(n)}}{n!} = \frac{1}{z(z-1)} \left[\frac{-(n+1)\delta - C(0)}{z^{n+1}} + \frac{(n+1)\delta - C'(0)}{z^n} - \sum_{k=0}^{n-2} \frac{C^{(n-k)}(0)}{(n-k)!} \frac{1}{z^{1+k}} \right]. \quad (8)$$

The solutions of eq.(1) for $x = 0$ are known in terms of hypergeometric functions. The values of two independent solutions above the cut in z running from 1 to $+\infty$ are

$$\begin{aligned} y_1^+(z) &= (1-z)^{\frac{1-\lambda_1}{2}} z^{\frac{1-\lambda_\nu}{2}} F\left(\frac{1-\lambda_1-\lambda_\infty-\lambda_\nu}{2}, \frac{1-\lambda_1+\lambda_\infty-\lambda_\nu}{2}, 1-\lambda_1; 1-z\right) \\ &\equiv -ie^{\frac{i\pi\lambda_1}{2}} t_1(z) \end{aligned} \quad (9)$$

$$\begin{aligned} y_2^+(z) &= (1-z)^{\frac{1+\lambda_1}{2}} z^{\frac{1+\lambda_\nu}{2}} F\left(\frac{1+\lambda_1+\lambda_\infty+\lambda_\nu}{2}, \frac{1+\lambda_1-\lambda_\infty+\lambda_\nu}{2}, 1+\lambda_1; 1-z\right) \\ &\equiv -ie^{-\frac{i\pi\lambda_1}{2}} t_2(z) \end{aligned} \quad (10)$$

and the asymptotic behavior at $z = +\infty + i\varepsilon$ is given by

$$\begin{aligned} Y_0^+(z) &= \begin{pmatrix} -ie^{\frac{i\pi\lambda_1}{2}} & 0 \\ 0 & -ie^{-\frac{i\pi\lambda_1}{2}} \end{pmatrix} \begin{pmatrix} t_1(z) \\ t_2(z) \end{pmatrix} \equiv \Lambda_1 \begin{pmatrix} t_1(z) \\ t_2(z) \end{pmatrix} \\ &\approx \Lambda_1 B_0 \begin{pmatrix} z^{\frac{1-\lambda_\infty}{2}} \\ z^{\frac{1+\lambda_\infty}{2}} \end{pmatrix} \equiv B_0^+ \begin{pmatrix} z^{\frac{1-\lambda_\infty}{2}} \\ z^{\frac{1+\lambda_\infty}{2}} \end{pmatrix}. \end{aligned} \quad (11)$$

Similarly below the cut we have

$$Y_0^-(z) = \Lambda_1^{-1} \begin{pmatrix} t_1(z) \\ t_2(z) \end{pmatrix} \approx \Lambda_1^{-1} B_0 \begin{pmatrix} z^{\frac{1-\lambda_\infty}{2}} \\ z^{\frac{1+\lambda_\infty}{2}} \end{pmatrix} \equiv B_0^- \begin{pmatrix} z^{\frac{1-\lambda_\infty}{2}} \\ z^{\frac{1+\lambda_\infty}{2}} \end{pmatrix}. \quad (12)$$

The explicit form of the matrix B_0 will not be relevant for the following but we give it in the Appendix for completeness.

The procedure of [9] is to compute the described monodromy around 0 and x exploiting the knowledge of the asymptotic behavior of the solution at infinity. The chosen contour which embraces 0 and x starts at $+\infty - i\varepsilon$ i.e. below the cut in z , reaches $1 - i\varepsilon$ then performing a 2π clockwise rotation it reaches $1 + i\varepsilon$, then it goes to $+\infty + i\varepsilon$ and finally through a 2π anticlockwise rotation at infinity reaches the initial point $+\infty - i\varepsilon$. It is shown in Figure 1. The unperturbed $x = 0$ monodromy matrix for such a transformation

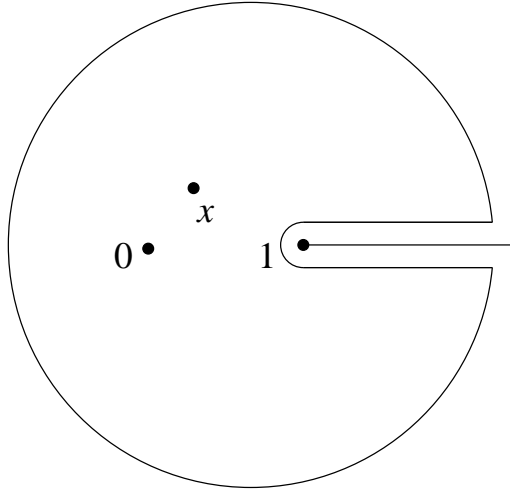


Figure 1: The monodromy contour

is

$$M_0 = B_0^+ \Lambda_\infty (B_0^-)^{-1} \quad (13)$$

where

$$\Lambda_\infty = \begin{pmatrix} e^{i\pi(1-\lambda_\infty)} & 0 \\ 0 & e^{i\pi(1+\lambda_\infty)} \end{pmatrix} \quad (14)$$

and one easily checks [9] that $\text{tr} M_0 = -2 \cos \pi \lambda_\nu$. We perform now the transformation

$$z(v, x) = \frac{v - \mathcal{C} - \mathcal{B}_1/v - \mathcal{B}_2/v^2 + \dots}{1 - \mathcal{C} - \mathcal{B}_1 - \mathcal{B}_2 + \dots} \quad (15)$$

where

$$\begin{aligned} \mathcal{C} &= xc_1 + x^2c_2 + x^3c_3 + \dots \\ \mathcal{B}_1 &= x^2b_{11} + x^3b_{12} + x^4b_{13} + \dots \\ \mathcal{B}_2 &= x^3b_{21} + x^4b_{22} + x^5b_{23} + \dots \\ \mathcal{B}_3 &= x^4b_{31} + x^5b_{32} + x^6b_{33} + \dots \\ &\dots\dots\dots \end{aligned} \quad (16)$$

We note that the number of coefficients c_k, b_{kj} appearing to order n is just n . Such a transformation is not one-to-one in the complex plane, but to each finite order n , given some $0 < r < 1$, for small x , $z(v, x)$ is one-to-one for $|v| \geq r$. In fact $z(v, x) = z(v_1, x)$ for $v \neq v_1$ is equivalent to

$$1 = -ww_1(\mathcal{B}_1 + \mathcal{B}_2(w + w_1) + \dots + \mathcal{B}_n(w^{n-1} + w^{n-2}w_1 + \dots + w_1^{n-1})) \quad (17)$$

where $w = 1/v$, $|w| \leq 1/r$ and $w_1 = 1/v_1$, $|w_1| \leq 1/r$. On $|w| = 1/r$ the r.h.s. goes uniformly to zero for $x \rightarrow 0$ and thus for small $|x|$ by Rouché theorem [17] eq.(17) is never satisfied for $|w| \leq 1/r$. The monodromy contour of Fig.1 lies in such a domain $|v| \geq r$ and from eq.(15) we have also $z(1, x) = 1$ and $z(\infty, x) = \infty$.

Under the transformation (15) Q_0 goes over to $Q_0(z(v))\left(\frac{dz}{dv}\right)^2 - \{z, v\}$ where $\{z, v\}$ is the Schwarz derivative of z w.r.t. v [18] and we shall determine $C(x)$ as to have to each order in x

$$\begin{aligned} & Q_0(z(v))\left(\frac{dz}{dv}\right)^2 - \{z, v\} \\ = & \frac{\delta_0}{v^2} + \frac{\delta}{(v-x)^2} + \frac{\delta_1}{(v-1)^2} + \frac{\delta_\infty - \delta_0 - \delta - \delta_1}{v(v-1)} - \frac{C(x)}{v(v-1)(v-x)} = Q(v) . \end{aligned} \quad (18)$$

The transformation of the solutions Y is [18]

$$Y_v(v) \equiv Y(z(v))\left(\frac{dz}{dv}\right)^{-\frac{1}{2}} . \quad (19)$$

At infinity z goes over to $(v - C)/(1 - C - B_1 - \dots)$ and $\left(\frac{dz}{dv}\right)^{-\frac{1}{2}}$ becomes a constant. Thus the power behaviors at infinity of the solutions (not the coefficients) are unchanged i.e. they are still of the form $v^{\frac{1 \mp \lambda_\infty}{2}}$. Then as far as the behavior of the solution at infinity is concerned the only thing that changes is the matrix B and thus B^+ and B^- and such a change is given by the right multiplication of B_0 by a diagonal matrix.

The main point in the treatment is that in computing the monodromy matrix M only the asymptotic behavior of the Y i.e. only the matrices B^+ , B^- intervene.

We shall have

$$B^+ = \Lambda_1 B_0 (1 + xD_1 + x^2 D_2 + \dots) \quad (20)$$

$$B^- = \Lambda_1^{-1} B_0 (1 + xD_1 + x^2 D_2 + \dots) \quad (21)$$

where D_n are diagonal matrices. It is now easily seen that the monodromy relative to the described contour equals the one of the unperturbed (i.e. $x = 0$) case; in fact

$$\begin{aligned} M &= B^+ \Lambda_\infty (B^-)^{-1} = \Lambda_1 B \Lambda_\infty B^{-1} \Lambda_1 \\ &= \Lambda_1 B_0 (1 + xD_1 + x^2 D_2 + \dots) \Lambda_\infty (1 + xD_1 + x^2 D_2 + \dots)^{-1} B_0^{-1} \Lambda_1 \\ &= \Lambda_1 B_0 \Lambda_\infty B_0^{-1} \Lambda_1 = B_0^+ \Lambda_\infty (B_0^-)^{-1} = M_0 . \end{aligned} \quad (22)$$

In particular the trace of the monodromy matrix for the contour embracing both 0 and x is again $-2 \cos \pi \lambda_\nu$.

Equating the coefficients of the power expansion in x of eq.(18) we find to order $n, n+1$ equations corresponding to the different powers s in the denominator

$$\frac{1}{v^s(v-1)} . \quad (23)$$

where $s = 2, \dots, n+2$. Thus we see that not only the explicit form of the matrix B_0 is irrelevant but also the explicit form of the solutions i.e. the hypergeometric functions (9,10) is not relevant in the computation but only their asymptotic behaviors (11,12) matter. To first order we have to fit the two terms in eq.(6) with the expansion of the l.h.s. of eq.(18) using the two parameters $c_1, C'(0)$.

$$A_2 c_1 = \begin{pmatrix} -2\delta_\nu \\ \delta_\nu + \delta_\infty - \delta_1 \end{pmatrix} c_1 = \begin{pmatrix} -2\delta - C(0) \\ 2\delta - C'(0) \end{pmatrix} \equiv N_2 \quad (24)$$

i.e.

$$c_1 = \frac{2\delta + C(0)}{2\delta_\nu}, \quad C'(0) = \frac{(\delta_\nu - \delta_0 + \delta)(\delta_\nu - \delta_\infty + \delta_1)}{2\delta_\nu} - C(0) \quad (25)$$

The value of c_1 will be useful later.

To second order we have to fit the three terms in eq.(7) finding three equations for the coefficients $b_{11}, c_2, C''(0)$

$$\begin{aligned} A_3 \begin{pmatrix} b_{11} \\ c_2 \end{pmatrix} &= \begin{pmatrix} -3 - 4\delta_\nu & 0 \\ 3(1 - \delta_1 + \delta_\infty) + \delta_\nu & -2\delta_\nu \\ \delta_\nu - \delta_1 - \delta_\infty & \delta_\nu + \delta_\infty - \delta_1 \end{pmatrix} \begin{pmatrix} b_{11} \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} -3\delta - C(0) + 3c_1^2\delta_\nu \\ c_1^2(\delta_1 - 2\delta_\nu - \delta_\infty) + 3\delta - C'(0) \\ -C''(0)/2 \end{pmatrix} \equiv N_3 . \end{aligned} \quad (26)$$

As c_1 and $C'(0)$ are known from the previous step, eq.(26) determines $C''(0)$ [15, 9]

$$\begin{aligned} C''(0) &= -\frac{(\delta_\infty + \delta_\nu - \delta_1)[C'(0) - 3\delta + c_1^2(2\delta_\nu + \delta_\infty - \delta_1)]}{\delta_\nu} \\ &\quad - \frac{(C(0) + 3\delta - 3c_1^2\delta_\nu)[3\delta_1^2 + 3\delta_\nu^2 + 3\delta_\infty(1 + \delta_\infty) + \delta_\nu(3 + 2\delta_\infty) - 3\delta_1(1 + 2\delta_\nu + 2\delta_\infty)]}{\delta_\nu(3 + 4\delta_\nu)} . \end{aligned} \quad (27)$$

In addition we can compute b_{11} and c_2 which will be useful in computing the third order. In the Appendix we report the explicit form of the matrix A_4 appearing in the third order computation and the value of $C'''(0)$.

We notice that the matrices A_2, A_3, A_4, \dots are nested matrices i.e. A_{n+1} is obtained from A_n by adding to the left a $n+1$ dimensional column and they are “lower triangular” i.e. $A_n(h, k) = 0$ for $k > h$. This is easily seen from the nature of the transformation (15). It is trivial to carry on the procedure to any order using eq.(8).

3 Comparison with the quantum conformal blocks

In [7, 15] the expansion of the conformal blocks up to the second order and in [5] the expansion up to the third order in x is given.

For the conformal blocks we have [5]

$$\mathcal{B} = \sum_{n=0}^{\infty} x^n \mathcal{B}^{(n)} \quad (28)$$

with

$$\mathcal{B}^{(0)} = 1, \quad \mathcal{B}^{(1)} = \frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)}{2\Delta} \quad (29)$$

and for the explicit expression of $\mathcal{B}^{(2)}$ and the rather long expression of $\mathcal{B}^{(3)}$ see [5]. Δ_j are the quantum dimensions which are related to the δ_j in classical limit $b \rightarrow 0$ by $\Delta_j = \delta_j/b^2$. Starting from $n = 2$ the $\mathcal{B}^{(n)}$ also contain the central charge c which for Liouville theory is given by

$$c = 1 + 6(b + b^{-1})^2. \quad (30)$$

The translation dictionary from the notation adopted in the present paper and the one adopted in [5] (which is not the same as the one of [7, 15]) is

$$\delta_\nu \rightarrow \delta, \quad \delta_0 \rightarrow \delta_2, \quad \delta \rightarrow \delta_1, \quad \delta_1 \rightarrow \delta_3, \quad \delta_\infty \rightarrow \delta_4. \quad (31)$$

The relation between the conformal block and the accessory parameter $C_L(x)$ of [7, 15] which is related to our $C(x)$ by $C(x) = xC_L(x)(1-x)$ is given by [3, 7, 8, 19]

$$C_L(x) = \frac{\partial}{\partial x} f_\delta \quad (32)$$

with

$$f_\delta = (\delta - \delta_1 - \delta_2) \log x + \lim_{b \rightarrow \infty} b^2 \log \sum_{n=0}^{\infty} x^n \mathcal{B}^{(n)} \quad (33)$$

defining the classical conformal blocks. Thus we have

$$[xC_L(x)]|_{x=0} = \delta - \delta_1 - \delta_2 \quad (34)$$

$$[xC_L(x)]'|_{x=0} = \lim_{b \rightarrow 0} b^2 \mathcal{B}^{(1)} \quad (35)$$

$$[xC_L(x)]''|_{x=0} = \lim_{b \rightarrow 0} b^2 (4\mathcal{B}^{(2)} - 2\mathcal{B}^{(1)^2}) \quad (36)$$

$$[xC_L(x)]'''|_{x=0} = \lim_{b \rightarrow 0} b^2 (3(6\mathcal{B}^{(3)} - 6\mathcal{B}^{(1)}\mathcal{B}^{(2)} + 2\mathcal{B}^{(1)^3})). \quad (37)$$

We note e.g. that in (28) we have $\mathcal{B}^{(3)} \sim (\frac{1}{b^2})^3$, $\mathcal{B}^{(2)} \sim (\frac{1}{b^2})^2$, $\mathcal{B}^{(1)} \sim (\frac{1}{b^2})$ so that heavy cancellations occur in the $b \rightarrow 0$ limit in the expressions (36,37). From eq.(35,36,37) we can compute the $C'(0) = [xC_L(x)]'|_{x=0} - [xC_L(x)]|_{x=0}$, $C''(0) = [xC_L(x)]''|_{x=0} - 2[xC_L(x)]'|_{x=0}$, $C'''(0) = [xC_L(x)]'''|_{x=0} - 3[xC_L(x)]''|_{x=0}$. We checked that such values agree with the result obtained in the previous section and in the Appendix.

4 Conclusions

In the present paper we extended the technique to compute classical conformal blocks developed in [9] to all order in the modulus. The procedure is iterative and very simple. We checked the results against the expression of the quantum conformal blocks available in the literature finding complete agreement. One could consider the extension of such a procedure to the computation of the quantum corrections near the semiclassical limit.

Appendix

As we mentioned in the text the explicit value of the matrix B_0 is not relevant for the computations. For completeness however we report it here below

$$B_0 = \begin{pmatrix} \frac{\Gamma(1-\lambda_1)\Gamma(-\lambda_\infty)}{\Gamma\left(\frac{1-\lambda_1-\lambda_\infty-\lambda_\nu}{2}\right)\Gamma\left(\frac{1-\lambda_1-\lambda_\infty+\lambda_\nu}{2}\right)} & \frac{\Gamma(1-\lambda_1)\Gamma(\lambda_\infty)}{\Gamma\left(\frac{1-\lambda_1+\lambda_\infty-\lambda_\nu}{2}\right)\Gamma\left(\frac{1-\lambda_1+\lambda_\infty+\lambda_\nu}{2}\right)} \\ \frac{\Gamma(1+\lambda_1)\Gamma(-\lambda_\infty)}{\Gamma\left(\frac{1+\lambda_1-\lambda_\infty+\lambda_\nu}{2}\right)\Gamma\left(\frac{1+\lambda_1-\lambda_\infty-\lambda_\nu}{2}\right)} & \frac{\Gamma(1+\lambda_1)\Gamma(\lambda_\infty)}{\Gamma\left(\frac{1+\lambda_1+\lambda_\infty+\lambda_\nu}{2}\right)\Gamma\left(\frac{1+\lambda_1+\lambda_\infty-\lambda_\nu}{2}\right)} \end{pmatrix}. \quad (38)$$

To third order using eq.(8) we have the equation

$$\begin{pmatrix} -12 - 6\delta_\nu & 0 & 0 \\ 12 - 5\delta_1 + \delta_\nu + 5\delta_\infty & -3 - 4\delta_\nu & 0 \\ -3\delta_1 + \delta_\nu - \delta_\infty & 3(1 - \delta_1 + \delta_\infty) + \delta_\nu & -2\delta_\nu \\ \delta_\nu - \delta_\infty - \delta_1 & \delta_\nu - \delta_1 - \delta_\infty & \delta_\nu + \delta_\infty - \delta_1 \end{pmatrix} \begin{pmatrix} b_{21} \\ b_{12} \\ c_3 \end{pmatrix} = N_4 \quad (39)$$

with

$$N_4 = \begin{pmatrix} (10b_{11}c_1 + 4c_1^3)\delta_\nu - 4\delta - C(0) \\ (4b_{11}c_1 + c_1^3)\delta_1 + (-6b_{11}c_1 - 3c_1^3 + 6c_1c_2)\delta_\nu - (4b_{11}c_1 + c_1^3)\delta_\infty + 4\delta - C'(0) \\ (-b_{11}c_1 + 2c_1c_2)\delta_1 - (b_{11}c_1 + 4c_1c_2)\delta_\nu + (b_{11}c_1 - 2c_1c_2)\delta_\infty - C''(0)/2 \\ -C'''(0)/3! \end{pmatrix}. \quad (40)$$

The value of $C'''(0)$ is given by

$$\begin{aligned}
C'''(0) = & ((-C(0) - 4\delta + 10b_{11}c_1\delta_\nu + 4c_1^3\delta_\nu)(-6\delta_1 + 6\delta_\nu - 6\delta_\infty))/(6(2 + \delta_\nu)) \\
+ & (-6\delta_1 + 6\delta_\nu - 6\delta_\infty)(-((-C(0) - 4\delta + 10b_{11}c_1\delta_\nu + 4c_1^3\delta_\nu) \times \\
& (12 - 5\delta_1 + \delta_\nu + 5\delta_\infty)))/(6(-3 - 4\delta_\nu)(2 + \delta_\nu)) \\
+ & (C'(0) - 4\delta - 4b_{11}c_1\delta_1 - c_1^3\delta_1 + 6b_{11}c_1\delta_\nu + 3c_1^3\delta_\nu - 6c_1c_2\delta_\nu + 4b_{11}c_1\delta_\infty + c_1^3\delta_\infty)/(-3 - 4\delta_\nu)) \\
+ & (-6\delta_1 + 6\delta_\nu + 6\delta_\infty)((-C(0) - 4\delta + 10b_{11}c_1\delta_\nu + 4c_1^3\delta_\nu)(-6\delta_1 + 2\delta_\nu - 2\delta_\infty))/(24\delta_\nu(2 + \delta_\nu)) \\
- & (C'''(0) + 2b_{11}c_1\delta_1 - 4c_1c_2\delta_1 + 2b_{11}c_1\delta_\nu + 8c_1c_2\delta_\nu - 2b_{11}c_1\delta_\infty + 4c_1c_2\delta_\infty)/(4\delta_\nu) \\
+ & ((6 - 6\delta_1 + 2\delta_\nu + 6\delta_\infty)(-((-C(0) - 4\delta + 10b_{11}c_1\delta_\nu + 4c_1^3\delta_\nu) \\
& (12 - 5\delta_1 + \delta_\nu + 5\delta_\infty)))/(6(-3 - 4\delta_\nu)(2 + \delta_\nu)) \\
+ & (C'(0) - 4\delta - 4b_{11}c_1\delta_1 - c_1^3\delta_1 + 6b_{11}c_1\delta_\nu + 3c_1^3\delta_\nu - 6c_1c_2\delta_\nu + 4b_{11}c_1\delta_\infty + c_1^3\delta_\infty)/ \\
& (-3 - 4\delta_\nu)))/(4\delta_\nu))
\end{aligned} \tag{41}$$

where $C(0), C'(0), C''(0), c_1, c_2, b_{11}$ are already known from the previous steps.

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